

# Contents

<b>Preface</b>	<b>1</b>
<b>1 Functional Programming with the Booleans</b>	<b>13</b>
1.1 Declaring the Datatype of Booleans . . . . .	13
1.1.1 Aside: type levels . . . . .	15
1.1.2 Constructors <code>tt</code> and <code>ff</code> . . . . .	15
1.1.3 Aside: some compiler directives . . . . .	15
1.2 First steps interacting with Agda . . . . .	16
1.3 Syntax declarations . . . . .	16
1.4 Defining boolean operations by pattern matching: negation . . . . .	18
1.4.1 Aside: space around operators and in files . . . . .	19
1.5 Defining boolean operations by pattern matching: and, or . . . . .	20
1.6 The if-then-else operation . . . . .	23
1.6.1 Some examples with if-then-else . . . . .	24
1.7 Conclusion . . . . .	25
1.8 Exercises . . . . .	26
<b>2 Introduction to Constructive Proof</b>	<b>27</b>
2.1 A First Theorem about the Booleans . . . . .	27
2.1.1 Types as formulas: the Curry-Howard isomorphism . . . . .	28
2.1.2 Proofs as programs: more Curry-Howard . . . . .	29
2.2 Universal Theorems . . . . .	30
2.2.1 Proving the theorem using pattern matching . . . . .	31
2.2.2 An alternative proof and implicit arguments . . . . .	32
2.2.3 Using holes in proofs . . . . .	33
2.3 Another Example, and More On Implicit Arguments . . . . .	34
2.4 Theorems with Hypotheses . . . . .	37
2.4.1 The absurd pattern . . . . .	38
2.4.2 An alternative proof . . . . .	39
2.4.3 Matching on equality proofs . . . . .	41
2.4.4 The rewrite directive . . . . .	42
2.5 Going deeper: Curry-Howard and constructivity . . . . .	43
2.6 Further examples . . . . .	47
2.7 Conclusion . . . . .	48

---

2.8 Exercises . . . . .	49
<b>3 Natural Numbers</b>	<b>51</b>
3.1 Peano Natural Numbers . . . . .	52
3.2 Addition . . . . .	53
3.2.1 Simplest theorems about addition . . . . .	55
3.2.2 Associativity of addition and working with holes . . . . .	59
3.2.3 Commutativity of addition and helper lemmas . . . . .	62
3.3 Multiplication . . . . .	64
3.3.1 Distributivity of multiplication and choosing the induction variable . . . . .	65
3.3.2 Commutativity of multiplication . . . . .	67
3.3.3 Associativity of multiplication . . . . .	68
3.4 Arithmetic Comparison . . . . .	69
3.4.1 Dotted variables . . . . .	72
3.4.2 An equality test . . . . .	73
3.5 Even/odd and mutually recursive definitions . . . . .	75
3.6 Conclusion . . . . .	76
3.7 Exercises . . . . .	76
<b>4 Lists</b>	<b>79</b>
4.1 The List Datatype and Type Parameters . . . . .	79
4.2 Basic Operations on Lists . . . . .	81
4.2.1 Length of a list . . . . .	81
4.2.2 Appending two lists . . . . .	81
4.2.3 Mapping a function over a list . . . . .	83
4.2.4 Filtering a list . . . . .	84
4.2.5 Removing an element from a list, and lambda abstractions .	85
4.2.6 Selecting the $n$ 'th element from a list, and the maybe type .	86
4.2.7 Reversing a list (easy but inefficient way) . . . . .	87
4.2.8 Efficiently reversing a list . . . . .	87
4.3 Reasoning about List Operations . . . . .	89
4.3.1 Distributing length over append . . . . .	89
4.3.2 Associativity of list append . . . . .	90
4.3.3 Length of filtered lists, and the with construct . . . . .	91
4.3.4 Filter is idempotent, and the keep idiom . . . . .	95
4.3.5 Reverse preserves list length . . . . .	98
4.4 Conclusion . . . . .	100
4.5 Exercises . . . . .	100
<b>5 Internal Verification</b>	<b>103</b>
5.1 Vectors . . . . .	103
5.1.1 The Vector Datatype . . . . .	104
5.1.2 Appending vectors . . . . .	105
5.1.3 Head and tail operations on vectors . . . . .	106
5.1.4 Using types to express properties of other vector operations	107

5.2	Binary Search Trees . . . . .	110
5.2.1	The <code>bst</code> module, and <code>bool</code> -relations . . . . .	110
5.2.2	The <code>bst</code> datatype . . . . .	114
5.2.3	Searching for an element in a binary search tree . . . . .	115
5.2.4	Inserting an element into a binary search tree . . . . .	116
5.3	Sigma Types . . . . .	117
5.3.1	Why Sigma and Pi? . . . . .	121
5.4	Braun Trees . . . . .	122
5.4.1	The <code>braun-tree</code> datatype, and sum types . . . . .	122
5.4.2	Inserting into a Braun tree . . . . .	124
5.4.3	Removing the minimum element from a Braun tree . . . . .	127
5.5	Discussion: Internal vs. External Verification . . . . .	129
5.6	Conclusion . . . . .	132
5.7	Exercises . . . . .	132
6	<b>Type-level Computation</b>	135
6.1	Integers . . . . .	135
6.1.1	The $\mathbb{Z}$ datatype . . . . .	136
6.1.2	Addition on integers . . . . .	137
6.1.3	Antisymmetric integer comparison . . . . .	139
6.2	Formatted Printing . . . . .	139
6.2.1	An unsuccessful attempt . . . . .	140
6.2.2	A working solution . . . . .	142
6.3	Proof by Reflection . . . . .	146
6.3.1	The datatype of representations, and its semantics . . . . .	147
6.3.2	The list simplifier . . . . .	148
6.3.3	Proving that the simplifier preserves the semantics . . . . .	151
6.4	Conclusion . . . . .	152
6.5	Exercises . . . . .	153
7	<b>Generating Agda Parsers with <code>grat</code></b>	155
7.1	A Primer on Grammars . . . . .	156
7.1.1	Derivations . . . . .	157
7.1.2	From derivations to syntax trees . . . . .	158
7.1.3	Regular-expression operators for grammars . . . . .	162
7.2	Generating Parsers with <code>grat</code> . . . . .	162
7.2.1	Compiling the emitted parsers . . . . .	164
7.2.2	Running the emitted executable . . . . .	165
7.2.3	Modifying the emitted code to process parse trees . . . . .	166
7.2.4	Reorganizing rules to resolve ambiguity . . . . .	167
7.3	Conclusion . . . . .	170
7.4	Exercises . . . . .	170
8	<b>A Case Study: Huffman Encoding and Decoding</b>	173
8.1	The Files . . . . .	174
8.2	The Input Formats . . . . .	175

8.3	Encoding Textual Input . . . . .	175
8.3.1	Computing word frequencies using tries . . . . .	177
8.3.2	Initializing a priority queue with Huffman leaves . . . . .	179
8.3.3	Processing the priority queue to compute a Huffman tree . .	182
8.3.4	Computing the code from the Huffman tree . . . . .	183
8.3.5	Encoding the input words using the computed code . . . . .	183
8.4	Decoding Encoded Text . . . . .	185
8.4.1	Code trees . . . . .	185
8.4.2	Computing the code tree . . . . .	185
8.4.3	Decoding the input . . . . .	187
8.5	Conclusion . . . . .	188
8.6	Exercises . . . . .	188
<b>9</b>	<b>Reasoning about Termination</b>	<b>189</b>
9.1	Termination Proofs . . . . .	189
9.1.1	Defining termination in Agda . . . . .	190
9.1.2	Termination of greater-than on natural-numbers . . . . .	192
9.1.3	Proving termination using a measure function . . . . .	193
9.1.4	Using termination proofs for well-founded recursion . . . . .	195
9.1.5	Nontermination with negative datatypes . . . . .	197
9.2	Operational Semantics for SK Combinators . . . . .	198
9.2.1	Mathematical syntax and semantics for combinators . . . . .	199
9.2.2	Defining the syntax and semantics in Agda . . . . .	201
9.2.3	Termination for S-free combinators . . . . .	203
9.2.4	Defining lambda abstractions using combinators . . . . .	207
9.3	Conclusion . . . . .	213
9.4	Exercises . . . . .	215
<b>10</b>	<b>Intuitionistic Logic and Kripke Semantics</b>	<b>217</b>
10.1	Positive Propositional Intuitionistic Logic (PPIL) . . . . .	218
10.2	Kripke structures . . . . .	222
10.3	Kripke semantics for PPIL . . . . .	225
10.4	Soundness of PPIL . . . . .	229
10.4.1	Monotonicity proof . . . . .	230
10.4.2	Soundness proof . . . . .	232
10.5	Completeness . . . . .	234
10.5.1	A universal structure . . . . .	234
10.5.2	Completeness and soundness with respect to $\mathbb{U}$ . . . . .	236
10.5.3	Concluding completeness and universality . . . . .	240
10.5.4	Composing soundness and completeness . . . . .	241
10.6	Conclusion . . . . .	247
10.7	Exercises . . . . .	247
	<b>Quick Guide to Symbols</b>	<b>249</b>
	<b>Commonly Used Keyboard Commands</b>	<b>251</b>

<b>Some Extra Emacs Definitions</b>	<b>253</b>
<b>References</b>	<b>255</b>
<b>Index</b>	<b>259</b>